Hirotaka Ebisui's Geometry Problems in Teaching Undergraduate Mathematics

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Abstract: *For a number of years one of the authors of this paper*[1] *has been collecting geometry problems created by the Japanese mathematician – Hirotaka Ebisui. In this paper we will make a brief analysis of some of these problems, examine their roots, and point out their value for the teaching of undergraduate mathematics.*

1. Introduction

Elementary geometry has quite a long and rich history. We can trace its origins in ancient times. The word *geometry* comes from the Greek word *geometrein* and has its source in two Greek words – *geo* which means *earth* and *metrein* meaning *measuring*. However, the real roots of elementary geometry can be traced much earlier in ancient civilizations of Egypt and Babylon. We know also that geometry existed in ancient Hindu and Chinese civilizations (see [2] and [4]).

In Egypt, geometry was a collection of rules or procedures obtained by experimentation, observation of analogies, sometimes guessing, and intuition. As we can easily guess it was knowledge full of errors and inaccuracies. However, at this time it was a sufficient enough knowledge for practical needs of the time. It was used mostly for land measuring, architecture and sometimes for carpentry works. In fact, the ancient Egyptian geometry (3000 BC to about 300 BC), is not considered as a science in our sense. It is just a collection of rules for calculations in land measuring, sometimes without any motivation or theoretical justification.

Babylonian geometry (2000 to 1600 BC) was more advanced than the geometry in Egypt. They knew a theorem known later as the Pythagoras theorem (about 1500 years before Pythagoras). In fact in literature (see [2]) we can find speculations that Babylonian mathematics had quite strong influence on later Greek mathematics.

Finally Greek geometry, starting from Thales of Miletus, was a deductive science. There we have geometric statements established on the base of deductions and formal proofs. This is the time when some of the most interesting developments in elementary geometry occurred. This is the time of the famous works of Thales of Miletus, Pythagoras and his disciples, Eudoxus, Hippocrates, Socrates, Plato and Euclid. The famous Euclid's *Elements of Geometry* is a summary of a large part of the mathematical knowledge of this time, and contains 13 books incorporating current, at this time, developments, as well as earlier works of Hippocrates (covered in Euclid's elements books I – IV), Pythagoreans (covered in books I-IV, VII, IX), Archytas (book VIII), Eudoxus (books V, VI, XII) and Theatetus (books X and XIII). It is important to notice that Elements do not contain all

[1] Hirotaka Ebisui is the author of the geometry problems presented in this paper. Mirek Majewski is the author of the text and images. All geometry images were created using Geometer's Sketchpad version 5.

mathematical knowledge of this period. Euclid himself wrote eight more advanced books on geometry. Ancient Greeks created a new approach to elementary geometry by using postulates and proving geometric facts by deductive reasoning from postulates.

A very interesting approach to mathematics, including geometry, had place in ancient China. The Chinese were always, and still are, very pragmatic. Their approach to mathematics was very algorithmic, and similar to the way how in our times we program our computers. A simple example taken from *Jiuzhang Suanshu* (English translation and commentaries [7]) demonstrates the Chinese way of dealing with mathematical problems:

Example Right-Angled Triangles

PROBLEM: Now, given a right-angled triangle, the lengths of its gou and gu are 3 chi and 4 chi respectively. Tell: what is the length of the hypotenuse?

SOLUTION: Add the squares of the gou and the gu, take the square root [*of the sum*] *giving the hypotenuse.*

This solution can be easily recognized as a verbally formulated algorithm:

```
Input gou, gu
A := gou^2+gu^2
H := sqrt(A)
Output H
```

Most of the solutions presented in [7] use the same approach to solving problems, i.e. by following specific algorithms.

2. Japanese geometry from the Edo period

Fig. 1 *Reconstructed sangaku from the Takemizuke shrine, Nagano prefecture, 450x200cm*

The Edo period in Japan, sometimes referred to as a renaissance in Japanese art and science, was also the time when a very specific form of mathematics flourished. This is mathematics that Japanese call *wasan* in order to distinguish it from the mathematics in the west known as *yosan*. Wasan problems were quite often painted or drawn on wooden tablets, called *sangaku,* and hung in Shinto or Buddhist temples. It was a way of thanking the gods for a moment of enlightenment while solving a mathematical problem, or a demonstration of the person’s mathematical problem solving skills. Some of the sangaku were just simple pieces of wood with some basic text and usually a picture if the content was related to geometry. Some other sangaku were prepared on large wooden boards, with many theorems, floral decorations and pictures. There were very small sangaku and huge ones a few meters long with a massive amount of information on them (see [4], [5] and [6]).

Many of the problems exposed on sangaku tablets were elementary geometry problems (see fig.1). However we can easily notice a fundamental difference between sangaku geometry and the western geometry. On sangaku tablets we have complex constructions of circles inside of other circles, ellipses, rectangles and other figures. Usually these are problems where tangency of objects plays major role. At the same time in western geometry we deal with crossing lines, triangles, angles, and sometimes circles.

Most of the geometry problems on sangaku tablets are very static in such a sense that we cannot move objects used there without changing their configuration and properties. For example a circle inscribed in a square or a triangle cannot be moved without changing the shape of the outer figure and the illustrated property. In many examples we deal with very complex figures where nothing can be moved. At the same time in western mathematics we deal with configurations of objects where there is quite a lot of freedom for moving these objects while the property, i.e. theorem, will be still valid. Figure 2 shows pictures of two geometric constructions. The construction on the left is a typical example of the western geometry. It is a construction of a tangent line to a circle from a given point *A*. Here both, the location of the point *A*, as well as location of the circle, and its size, can be changed and the line constructed this way will still be tangent to the circle. The construction on the right illustrates one of the sangaku problems where we have to calculate radius of one of the circles depending on the radius of the other circle. Here nothing can be moved.

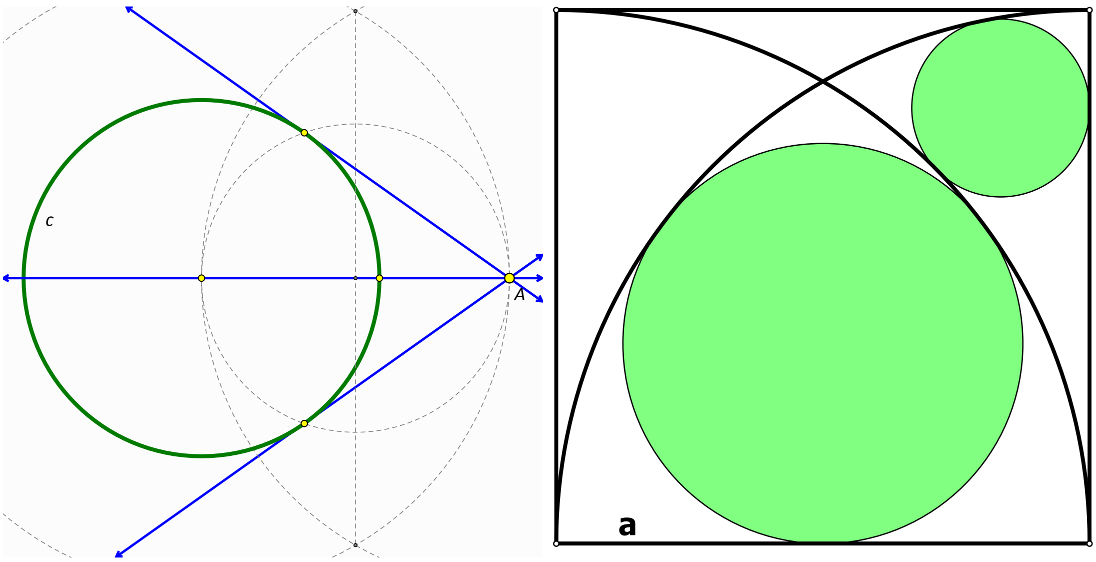


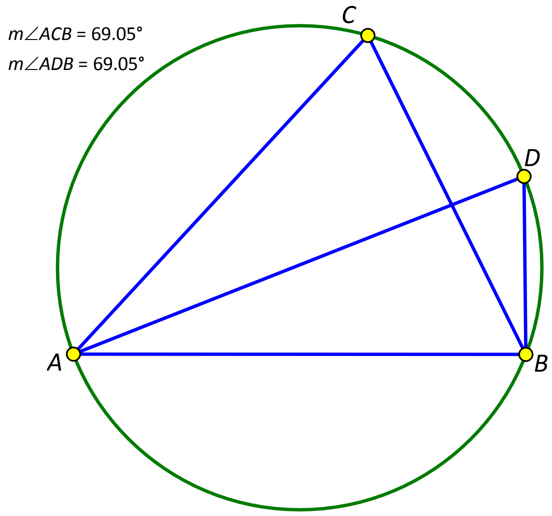
Fig. 2 *Two geometric constructions: application of the Thales theorem (left) and a sangaku problem (right).*

3. Dynamic approach to elementary geometry

Invention of computers and easy-to-use computer software opened a new era in elementary geometry. Now, anybody can open GeoGebra, Geometer’s Sketchpad, Cabri or any other geometry software, develop his own geometry problems, experiment with them and obtain facts that can be verified with the computer without a formal proof of them. This leads to a vast number of interesting discoveries in elementary geometry. Let us look for a moment at a very simple example that can be obtained by a high school student while experimenting with any dynamic geometry software, e.g. Geometer’s Sketchpad or Cabri.

Example – students' explorations

Basic experiments in Sketchpad with circles and a few lines, or triangles inscribed in a circle, lead to "discoveries" of many interesting properties. These properties usually are well known and their proofs are published in geometry textbooks, but students can discover them on their own, check them and later prove them or search a textbook for the proof. This builds some special interest in elementary geometry.



Having a circle and two angles based on a common arc (here the arc AB and angles ∡C, ∡D), we can easily notice that the two angles are equal. Any dynamic geometry software, where measuring of angles is available, will confirm this fact.

From here we can get a conclusion that two triangles, with a common base, inscribed in the same circle, have equal angles opposite the base (here ∡C=∡D).

We can also easily find out that if one side of the triangle passes through the center of the circle then the opposite angle is 90°.

Some of the discoveries of high school students go much beyond these examples. This opens a completely new era in elementary geometry – discovering facts by experimenting and later proving them.

4. Hirotaka Ebisui and his geometry problems

Hirotaka Ebisui is a Japanese mathematician, known as a person who is fascinated with the plane, elementary and projective geometry. In the last few years he produced a number of geometric constructions that illustrates a mathematical property. It is very difficult to evaluate how many such problems Hirotaka Ebisui has created. For the last few years one of the authors (M.M) has been collecting all files created by Hirotaka and combining them in one large document (about 220 pages), and each page containing 3 to 4 constructions. This makes about 800 constructions. There are still more Hirotaka's files that were not included in this collection. This author has also been recreating many of Hirotaka's constructions using Geometer's Sketchpad, sometimes with Geometry Expressions. Each of them turns out to be a true fact.

There is no doubt that Hirotaka's constructions have roots in sangaku problems. We can easily notice that many of his constructions remind us of the pictures that we have seen on sangaku tablets from the Edo period and later. The major difference is that his works are always constructions, while the ones on sangaku tablets were always drawings. We can also see that his constructions combine the western geometry point of view with the dynamic geometry approach.

There one thing more that is worth to mention. His attitude while developing his constructions is very similar to the attitude that accompanied the old sangaku masters. From time to time on his constructions we can see a sentence where he thanks the Goddess of Mathematics or the Sun, or Earth for the joy of a new discovery. Similarly to the old masters of sangaku he frequently sends his works to friends with a text 'this is my theorem, please enjoy'.

Fig. 3 *Hirotaka Ebisui with Professor Gunter Weiss from TU in Dresden*

The original Hirotaka's Ebisui documents are usually very hard to understand and as a result only very few people are able to get a sense from them and appreciate his works.

One day Prof. Gunter Weiss from Technical University in Dresden, in Germany, wrote to me[2]:

"*I personally am convinced that Hirotaka is a genius with an incredible instinct for extending and generalizing elementary geometric problems in an interesting and surprising way. He stands in the tradition of the so-called "Japanese Temple Geometry" both, concerning the elementary geometric topics as well as in his behavior not to publish his findings other than showing them to friends with the words "I found a new theorem, please enjoy".*

He has never had a "scientific pupil" and his work is not "mathematical mainstream" at all. But it is so fascinating!

All this together makes me a true lover of his work, which expresses Japanese philosophy and tradition AND the beauty of pure simple geometry, which has no other aim than Japanese cherry blossoms.

He cannot be forced to work on a certain problem one proposes to him, but he is like a never stopping fountain, when one listens to him and gets stimulated to work on his findings. He very seldom proves his "theorems" and seems to be quite content, if his graphics-software constantly shows the incidences if zoomed. But he also knows the "geometer's toolbox" for elementary geometry very well (Desargues' and Pappus' and Brianchon' theorems, angle at circumference, power of a point with respect to a circle ...) and sometimes adds a proof, if asked for that.

I believe that his findings really are new and that nobody else dealt with this material before him. So the findings (- I call them findings and not theorems -) are due to him."

[2] Personal communication with M. Majewski, published with permission of Prof. Gunter Weiss

5. Discussion of selected Hirotaka's geometry problems

In this chapter we will discuss selected Hirotaka's Ebisui constructions with a simple explanation of what is there, and what should be proved. In order to make these problems a bit more similar to the sangaku problems from the Edo period some fills were added. We selected only those problems that are simple enough to be understood by an average high school student.

Before we proceed with the examples, it would be important to explain the convention used while developing the enclosed pictures. We used a specific color scheme that cannot be seen in black-and-white printing. Therefore, we also vary the thickness of lines and points.

Medium thick lines, green, are the starting point of the construction. Thin lines, usually blue, sometimes dashed are the construction lines. In many cases, if these lines are not needed any more, we hide them. However, sometimes we leave them in order to show more clearly how the construction was done. The thick, red lines, are the final ones that usually carry some property to be proved or are part of the final object, e.g. a triangle, rectangle, etc., with specific properties to be proved.

Usually we hide all points that do not matter for the whole construction. We leave only selected points to emphasize some important features, e.g. point on a segment, on a circle, etc. These points are always small and with light blue filling. The medium size points with yellow filling are those construction points that in GSP file, or in GSP online applet, can be moved in order to check what will happen with the property to be proved.

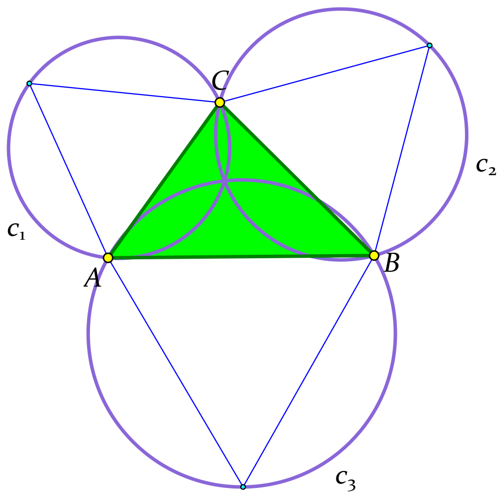
Finally, fills do not follow the above convention. We added them in order to make these constructions similar to the Edo period sangaku pictures.

The numbers for each problem are the original numbers used by Hirotaka Ebisui. In order to avoid some confusion there is one thing that we should mention here. It happens from time to time that a few of his constructions may carry the same number.

Some of the problems presented here are a bit more complicated than the remaining. Therefore, we illustrate them using two pictures. The first picture shows the initial stage of the construction, which is with what we started with and the second picture, shows the final construction.

Example 1: From any triangle to the equilateral triangle (HI-A1-019-020)

This example is one of the most recent constructions created by Hirotaka. It is simple enough to use as a high school problem, and the proof can be done by a high school student.

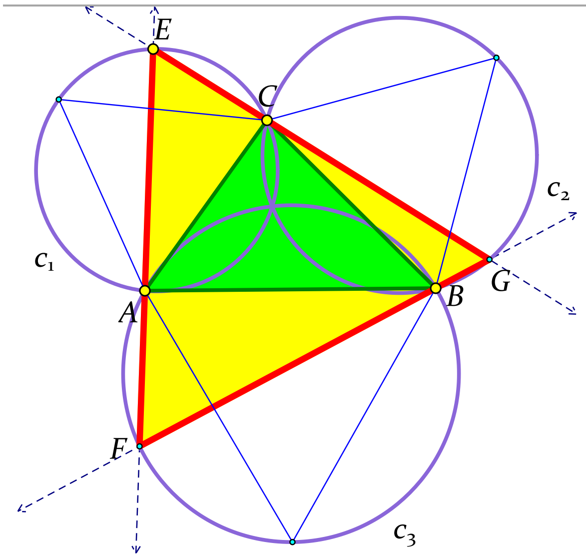


We start by drawing any triangle, here this is ΔABC.

Then we construct equilateral triangles with bases AB, BC and CA.

Each of these new triangles we inscribe in a circle, here these are circles c_1, c_2 and c_3.

This way we get a construction shown to the left of this text.



Select a new point on the circle c_1, it will be point E.

Draw a line through points E and A, and another line through points E and C.

Points of intersection of these new lines with circles c_2 and c_3 mark as F and G.

Draw another line passing through points F and G.

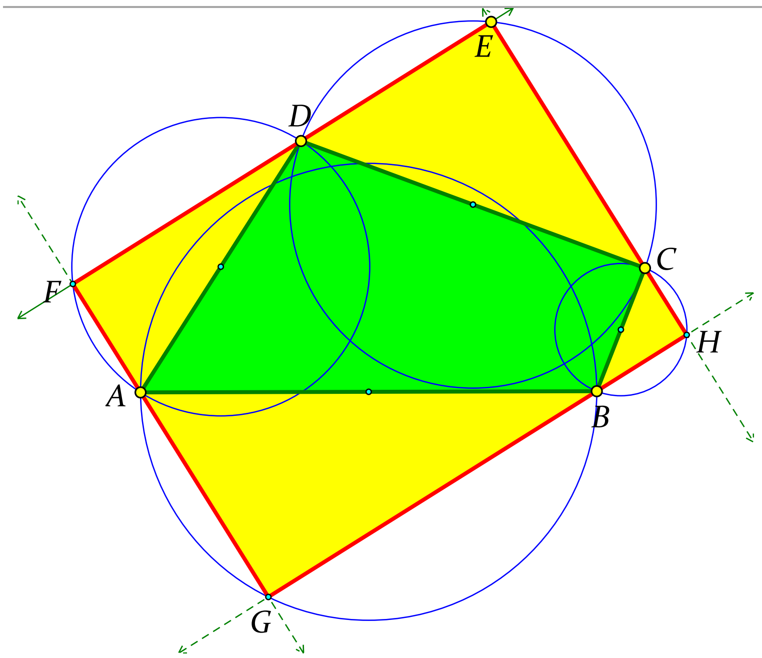
This way we got a new triangle ΔEFG. Mark its edges using a thick line (see picture to the left).

Prove that ΔEFG is an equilateral triangle.

The proof of this fact is very straightforward. Each small triangle is equilateral. Therefore, all its angles are 60°. In such case angles ∡*E*, ∡*F* and ∡*G* are also equal to 60°. This makes triangle *ΔEFG* equilateral.

The example can be expanded in a few ways and in a few directions. In the next example we show its version for an arbitrary quadrilateral.

Example 2: From any quadrilateral to a rectangle (HI-AI-020)



We start by drawing an arbitrary quadrilateral with four vertices, A, B, C and D.

We construct four circles each with one side of the quadrilateral as a diagonal.

On one of these circles we select a new point, here it is point E.

In exactly the same way as in the previous example we draw lines connecting E with points D and C obtaining this way two new points F and H

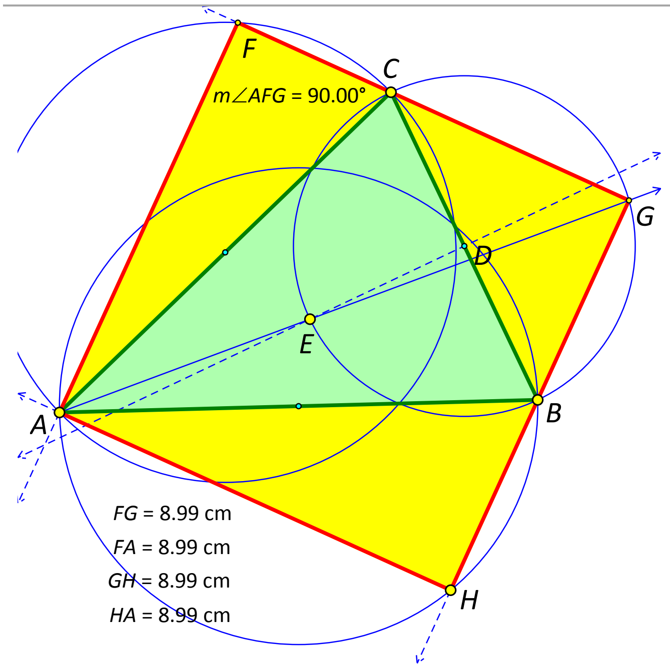
We draw two more lines connecting A with F and B with H. These lines intersect in the point G.

This is all, and now our task is to ***prove that quadrilateral EFGH is a rectangle.***

Proof of this fact is simple if we remember that if a triangle is inscribed in a circle, and one of its sides is the diagonal of the circle, then the angle opposite it has 90°. Therefore, we got a quadrilateral with three angles equal to 90°. This makes the fourth angle ∡*G* also 90° and proves that the quadrilateral *EFGH* is a rectangle.

Example 3: From a triangle to a square (HI-AI-028)

This is another nice example from the series of Hirotaka's problems where starting from an arbitrary triangle we construct an equilateral triangle or a rectangle.



We start from an arbitrary triangle ΔABC. On each side of it we construct a circle with the triangle side as diagonal.

From the center of one of the circles (here D) we construct a line perpendicular to the triangle side (here CD⊥DE). This way we obtain the intersection point E.

From the point A we draw a ray through E, getting this way the point G.

We draw rays GC and GB obtaining this way two new points F and H.

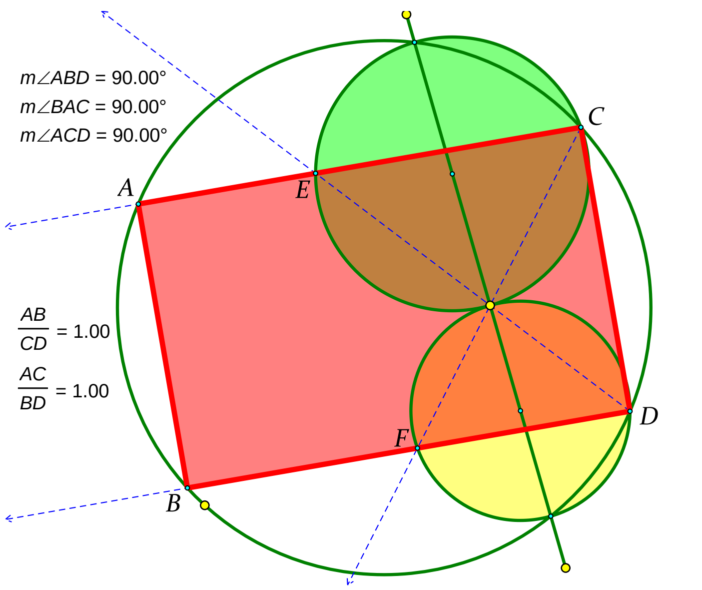
Finally we draw the segment AH.

Now, we can check in a computer program that FG=FA=GH=AH, and one of the angles is 90°.

Prove that the figure AFGH is a square.

A reasonably easy and completely elementary proof of this fact we leave to our readers.

Example 4: From three circles to a rectangle (HI 001)



Create a large circle, a segment crossing it, select a point on the segment and construct two small circles passing through the selected point and points of intersection of the segment with the large circle. Centers of the small circles should be located on the segment.

Construct rays from C and D passing through the point of tangency of both small circles. We will obtain two new points E and F.

Construct rays CE and DF, and intersection points A and B on the large circle.

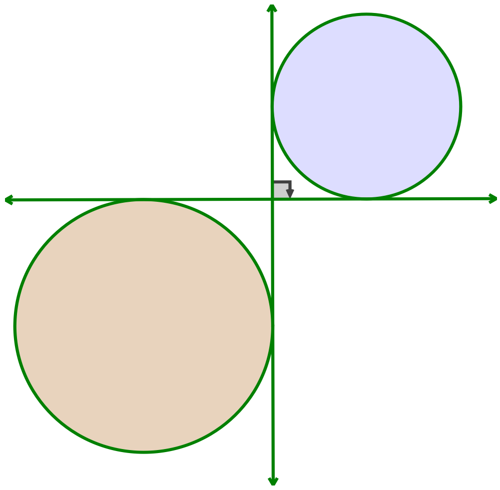
Connect points ABCD and fill the area inside.

The objective of this problem is to ***prove that figure ABCD is a rectangle.***

In GSP we can measure angles ∡ABD, ∡BAC and ∡ACD. GSP shows that each of them is 90°. We can easily check the proportions AB/CD and AC/BD and GSP shows that each of them is equal to 1.

These properties hold if we move the segment, change size of any circle as long as the segment intersects with the large circle and radius of any circle is strictly larger than 0.

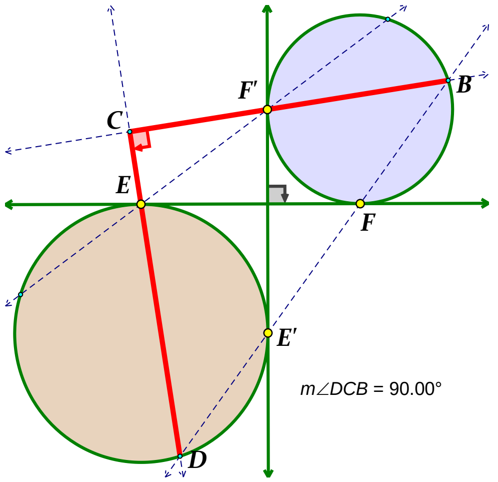
Example 5 Two perpendicular lines and two tangent circles (HI 004)



This is one of the nicest problems in Hirotaka files. It is repeated in Hirotaka files in a few different versions.

The starting point of this problem are two perpendicular lines, and two circles tangent to both lines and located in the opposite quarters of the plane.

Construct lines connecting points of tangency, each on different circle. You will obtain lines EF' and FE' as well as two new points B and D.



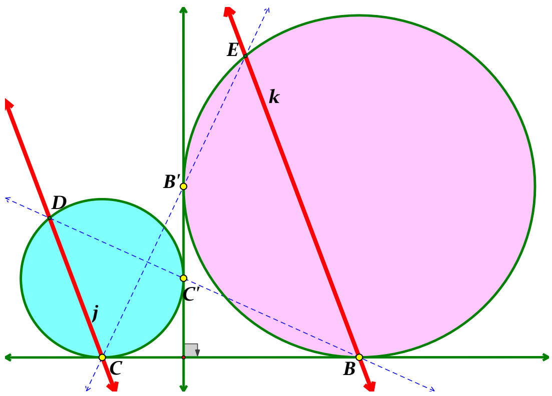
Now construct lines BF' and DE obtain the intersection point C, hide the lines, and draw segments CB and CD.

The objective of this problem is to prove that segments CB and CD are perpendicular.

GSP evaluated this angle as 90 degrees and its value does not change if we change the size of any of these two circles.

A reasonably easy, and completely elementary, proof based on the properties of angles inscribed in a circle we leave to our readers.

Constructions of the following examples are slightly simpler. Therefore, we present them in a short form. We leave proofs of them to the readers.



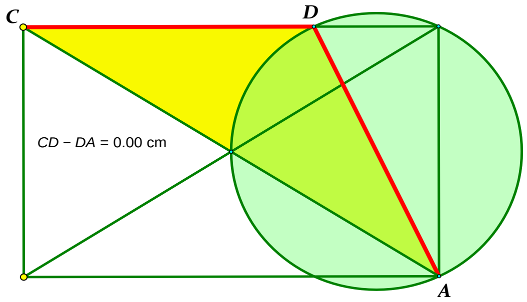
Example 6: HI 028

This problem is just another version of the problem HI 004. Here both tangent circles are on the same side of one of the two perpendicular lines.

Connect the tangency points to obtain the two dashed lines.

Finally draw lines passing through points of intersection of the dashed lines with circles (lines k and j).

The objective is to ***prove that lines k and j are parallel.***



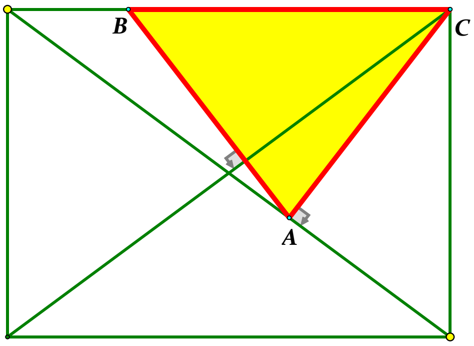
Example 7: HI 122

This is another simple and nice problem.

Start with a rectangle and a circle. Two adjacent points of the rectangle are on the circle.

One of the intersection points of the rectangle with the circle label as D, construct triangle ADC.

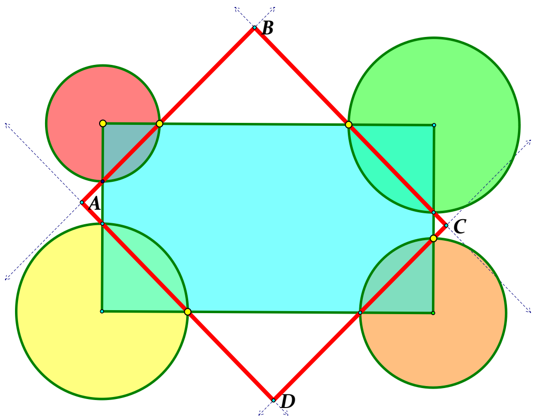
The objective is to ***prove that the ADC is an isosceles triangle****.*



Example 8: HI 290

Start with a rectangle and its diagonals. From one of the vertices construct a segment perpendicular, here CA, to one of the diagonals, and from its other end, here point A, draw another segment perpendicular to the other diagonal. Extend the new segment until it will contact the side of the rectangle, here it is AB.

The objective is to ***prove that CAB is an isosceles triangle****.*



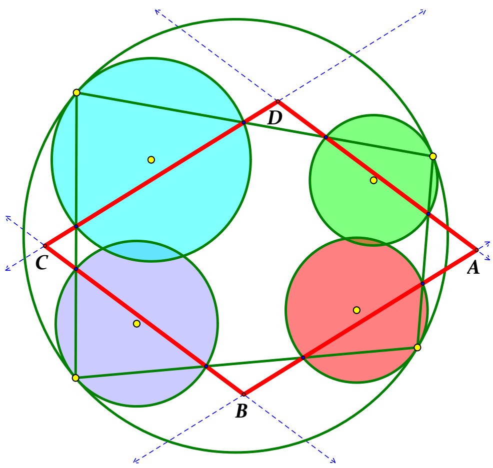
Example 9: HI 222

A rectangle and four circles are given. Each vertex of the rectangle is a center of one circle.

For each of the circles draw a line passing through points of intersection of the circle with sides of the rectangle.

Construct a polygon ABCD using the four new lines and their intersection points.

The objective is to prove that the polygon ABCD is a rectangle.

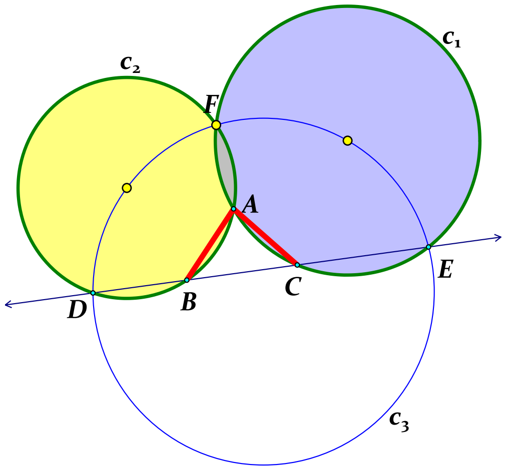


Example 10: HI 287

This problem is in some sense similar to the problem HI 222. Start with a large circle and four smaller circles inside the large circle and tangent to it. Create a polygon connecting the four points of tangency.

For each small circle draw a line passing through points of intersection of the circle with the sides of the polygon. Use these lines to create a new polygon, on the picture it is ABCD.

The objective is to ***prove that ABCD is a parallelogram****.*



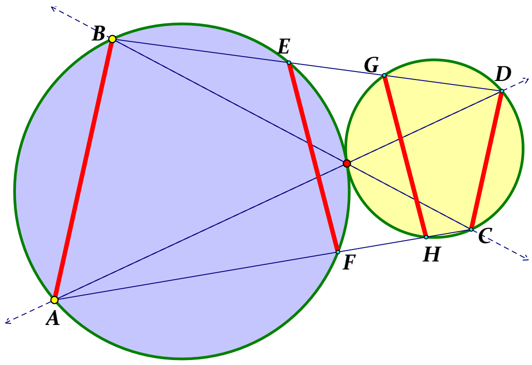
Example 11: HI 069

This is Hirotaka's favorite problem. He published it on one the problem solving portals in Japan and he got quite enthusiastic response.

Two intersecting circles c_1 and c_2 are given. Through one of the points of intersection, here point F, and centers of the circles construct a new circle c_3. Points of intersection c_3 with two existing circles label as D and E.

Draw line passing through points D and E. The line intersects the two circles c_1 and c_2 in two new points B and C. Connect points A with B and A with C.

The objective is to ***prove that |AB|=|AC|***



Example 12: HI 386

There are given two tangent circles. On one of them choose arbitrarily two points A and B.

Construct two lines each one passing through the tangency point of circles and points A and B respectively. Label points of intersection of the line with the other circle as D and C. Draw segments BD and AC.

Draw segments connecting opposite points on the edge of each circle, i.e. AB, EF, GH, and DC.

The objective is to ***prove that AB||DC and EF||GH****.*

6. Conclusions

We could continue for many hours, days, months, or even years, explorations of Hirotaka's constructions. Many of the Hirotaka's problems can be used by teachers as exercises in problem solving in regular geometry classes. Some other can be used with mathematically gifted high school, as well as university, students.

What is really important to point out here, is that all his problems are a valuable source of explorations in high school geometry topics: properties of circles, triangles, triangles inscribed in circles, perpendicularity of lines and segments, parallelism of lines, and many other elementary geometry topics that we teach every day. We have there also applications of basic theorems and sometimes less known facts in elementary geometry.

In some more advanced Hirotaka's problems we can find applications, or extensions, of Napoleon, Morley, and many other famous theorems in elementary or projective geometry. Some other of his problems can be a starting point for advanced research in elementary geometry. Many of them should be considered as new valuable theorems in modern geometry.

7. References

[1] *A history of Japanese Mathematics*, D. E. Smith & Y. Mikami, Cosimo, Cosimo Classics series, ed. 2007.

[2] *Euclidean and Non-Euclidean Geometries – development and history*, M. J. Greenberg, w. H. Freeman and Company, New York, 1993.

[3] *Geometry Revisited*, H.S.M. Coxeter and S.L. Greitzer, The Mathematical Association of America, New York, 1967.

[4] *Japanese Temple Geometry Problems*, H. Fukugawa and D. Pedoe, Charles Babbage Research Foundation, Winnipeg, Canada, 1989.

[5] *Japanese Temple Geometry*, T. Rothman, Scientific American, 30, 2003.

[6] *Sacred Mathematics – Japanese Temple Geometry*, H. Fukagawa & T. Rothman, Princeton University Press, ed. 2008.

[7] *The Nine Chapters of the Mathematical Art,* S. Kangshen, J. Crossley, A. W.C. Lun, Oxford University Press, Beijing, 1995.

Internet resources

[I1] *Hirotaka files at* `http://mirek.majewscy.net/hirotaka_files/`

[I2] *History of Greek Geometry at* `http://en.wikipedia.org/wiki/History_of_geometry`

[I3] *Hirotaka Ebisui Blog at* `http://aitoyume.de-blog.jp/`

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